“Cosmologists are often in error, never in doubt!”

Landau, Zel’dovich
2. Gravitational lenses

6. GL MODELS; 6.1. Axially symmetric lens models:

\[ p = \rho \frac{kT}{m}, \]  
\[ m \sigma^2 = kT. \]  
\[ \frac{dp}{dr} = -G M(r) \rho / r^2, \]  
\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho, \]  
\[ \rho(r) = \frac{\sigma^2}{(2\pi G r^2)}. \]  
\[ v_{\text{rot}}^2(r) = \frac{G M(r)}{r} = 2 \sigma^2 = \text{Cte}. \]  
\[ \Sigma(\xi) = \frac{\sigma^2}{(2 G \xi)}. \]

2. Gravitational Lenses: 6. GRAVITATIONAL LENS MODELS:

6.1. Axially symmetric lens models; 6.1.3. The SIS lens model:

A simple model for the mass distribution in galaxies assumes that the stars and other mass components behave like particles of an ideal gas, confined by their combined, spherically symmetric gravitational lens potential. The equation of state of the ‘particles’, henceforth called stars for simplicity, takes the form (Narayan and Bartelmann 1997)

\[ p = \rho \frac{kT}{m}, \]  
where \( \rho \) and \( m \) are the mass density and the mass of the stars. In thermal equilibrium, the temperature \( T \) is related to the one-component (observable) velocity dispersion \( \sigma \) of the stars in the galaxy through

\[ m \sigma^2 = kT. \]  

The temperature, or equivalently the velocity dispersion, could in general depend on radius \( r \), but it is usually assumed that the stellar gas is isothermal, so that \( \sigma \) is constant across the galaxy. The equation of hydrostatic equilibrium then gives

\[ \frac{dp}{dr} = -G M(r) \rho / r^2, \]  
with

\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho, \]

where \( M(r) \) is the mass interior to radius \( r \). A particularly simple solution of the previous equations is

\[ \rho(r) = \frac{\sigma^2}{(2\pi G r^2)}. \]  
This mass distribution is called the \textit{singular isothermal sphere}. 

2. Gravitational Lenses: 6. GRAVITATIONAL LENS MODELS:
6.1. Axially symmetric lens models; 6.1.3. The SIS lens model:
Since $\rho(r) \propto r^{-2}$, the mass $M(r)$ increases $\propto r$, and therefore the rotational velocity of test particles in circular orbits in the gravitational potential is

$$v_{\text{rot}}^2(r) = \frac{G M(r)}{r} = 2 \sigma^2 = C_{\text{te}}.$$  \hspace{1cm} (6.20)

The flat rotation curves of galaxies are thus naturally reproduced by this model (see the above figure). [This form of the density law could also have been retrieved assuming that the gradient of the Newtonian potential is $dU/dr = v^2/r$ and by straight application of the Poisson equation in spherical coordinates, we find that $\rho(r) = v^2/4\pi Gr^2$.]
2. Gravitational Lenses

6. GL MODELS; 6.1. Axially symmetric lens models:

\[ \alpha(\xi) = \alpha_o = 4 \pi \sigma^2 / c^2, \]  
\[ \theta_{A,B} = \theta_s \pm \alpha_o, \]  
\[ \theta_E = \alpha_o, \]  
\[ \mu_E = 4 \theta_E / d\theta_s. \]  

\[ \mu_A = 1 + (\theta_E / \theta_s), \]  
\[ \mu_B = 1 - (\theta_E / \theta_s). \]  
\[ \mu_T = \mu_A - \mu_B = 2 \theta_E / \theta_s. \]  
\[ \mu_A = 1 + (\theta_E / \theta_s). \]  

2. Gravitational Lenses: 6. GRAVITATIONAL LENS MODELS:

6.1. Axially symmetric lens models; 6.1.3. The SIS lens model:

Upon projecting the mass of the spherical model with volume density \( \rho(r) \) along the line-of-sight, we easily obtain the expression of the surface mass density \( \Sigma(\xi) = \int \rho(r=\sqrt{\xi^2+s^2}) ds \)

\[ \Sigma(\xi) = \sigma^2 / (2 \pi \xi), \]  
where \( \xi \) is the distance from the center of the two-dimensional profile. Referring to Eq. (4.5) and since \( M(\xi) = 2 \pi \int \Sigma(\xi') \xi' d\xi' \), we obtain the expression of the deflection angle

\[ \alpha(\xi) = \alpha_o = 4 \pi \sigma^2 / c^2, \]  
see the corresponding ray tracing and bending angle diagrams.

For \( |\theta_s| \leq \alpha_o = \alpha_o (D_{ds}/D_{os}) \), the solutions of the one-dimensional SIS lens equation are then \( \theta_{A,B} = \theta_s \pm \alpha_o \)

in accordance with Eq. (4.3). From these equations, we may directly infer that an observer located on the symmetry axis (i.e. for \( \theta_s = 0 \); see \( O_1 \) in the ray tracing diagram) will also see in this case an Einstein ring, the latter one being characterized by the angular radius

\[ \theta_E = \alpha_o, \]  
and that the magnification of this ring amounts to

\[ \mu_E = 4 \theta_E / d\theta_s. \]
2. Gravitational Lenses: 6. GRAVITATIONAL LENS MODELS:

6.1. Axially symmetric lens models; 6.1.3. The SIS lens model:

This last value for the magnification is found to be twice as large as that for the point mass lens; the reason being that the angular thickness of the SIS Einstein ring is $2 \theta_s$ (see the combined bending angle diagram), i.e. it is twice as large as that in the point mass case.

As the observer moves away from the symmetry axis (cf. $O_2$ in the ray tracing diagram), the Einstein ring also here breaks up in two images with an angular separation $\Delta \theta = 2 \theta_E$ (see the combined bending angle diagram). As long as $\theta_s \leq \theta_E$, we easily find by means of Eqs. (4.6) and (6.23) that the (positive and negative) magnification of the two images is

\[ \mu_A = 1 + \left( \frac{\theta_E}{\theta_s} \right), \quad (6.26a) \]

and

\[ \mu_B = 1 - \left( \frac{\theta_E}{\theta_s} \right). \quad (6.26b) \]

The net total magnification affecting the two images is thus

\[ \mu_T = \mu_A - \mu_B = 2 \left( \frac{\theta_E}{\theta_s} \right). \quad (6.27) \]

For $\theta_s > \theta_E$, the observer only sees one image (cf. $O_3$ in the ray tracing and combined bending angle diagrams) and its magnification is given by

\[ \mu_A = 1 + \left( \frac{\theta_E}{\theta_s} \right). \quad (6.28) \]

We see that $\mu_A \to 1$ when $\theta_s$ increases to large values. We recall that the SIS lens model constitutes a good first approximation to simulate the lensing properties of real galaxies over a large range of impact parameters and that it is therefore often being used for estimating lensing probabilities (see section 7). The most serious shortcoming of this model is its too
2. Gravitational lenses

6. GL MODELS; 6.1. Axially symmetric lens models:

Ray tracing diagram for the SIS lens model.
2. Gravitational lenses

6. GL MODELS:
6.1. Axially symmetric lens models:

Combined bending angle diagram (b) and resulting lensed images produced for the circular source S by a SIS lens model (a, c and d; see text).

2. Gravitational Lenses; 6. GRAVITATIONAL LENS MODELS:
6.1. Axially symmetric lens models; 6.1.3. The SIS lens model:
large deflection for light rays passing near to the center of the galaxy. However, for statistical purposes, this is not too serious since small values of the impact parameter $\xi$ do occur very seldomly. Since for real (finite) galaxies, the deflection angle obviously tends to zero as the impact parameter gets large, one may naturally introduce truncated singular isothermal sphere models or even more complex ones, as shown in the next section.
2. Gravitational lenses

6. GL MODELS:

6.1. Axially symmetric lens models:

6.1.4. The spiral galaxy lens model: If we combine Eq. (4.5) with Eq. (5.10) that characterizes the mass distribution of the disk of a spiral galaxy, seen face on, we may easily construct the resulting ray tracing diagram (cf. the above figure) and the combined bending angle diagrams (see next figures) in order to understand the formation of multiple lensed images due to this somewhat more complex deflector model. From these diagrams, we directly see that whenever the observer is located between the lens and the focal point caused by the inner part of the disk (cf. $O_1$ in the corresponding figures), we have $\Sigma_0 < \Sigma_c$, where $\Sigma_c$ represents the critical surface density defined by Eq. (6.5) and $\Sigma_0$ the central surface mass density of the spiral galaxy (cf. Eq. (5.8)). In this case, the observer only sees one single image of the distant source. These properties will be better understood after reading section (6.1.5).
2. Gravitational lenses

6. GL MODELS:

6.1. Axially symmetric lens models:

Combined bending angle diagrams (b and c) and resulting lensed images produced for the circular source S by a spiral galaxy lens model (a, d and e; see text).

Numerical simulations Caus2.exe and/or Caustics.exe !!!

2. Gravitational Lenses:

6. GRAVITATIONAL LENS MODELS:

6.1. Axially symmetric lens models:

6.1.4. The spiral galaxy lens model:

However, when the observer is located behind the focal point \( (\Sigma_0 > \Sigma_c) \), he may see either 3 images (cf. \( O_2 \) in the corresponding figures) or one single image (cf. \( O_3 \)), depending on whether he is located within, or outside, the caustic line. This caustic is very well seen in the ray tracing diagram as the envelope curve formed by the deflected light rays, which are tangent by pairs, originating from a very distant point source. Mathematically, the caustic corresponds to the solution of Eq. (4.6) for the case when the magnification of one of the lensed images gets infinitely large \( (i.e. \mu_i \rightarrow \infty) \). We will see in section 6.2. that caustics constitute generic features characterizing realistic lens models and that, very generally speaking, an observer crossing a caustic (cf. going from the positions \( O_2 \) to \( O_3 \)) always sees two of the lensed images approaching each other, getting very bright while they merge, and then totally vanishing when the observer gets on the other side of the caustic. For the case of the spiral galaxy lens model, this sequence of events may easily be understood while changing the position of the source in the bending angle diagram illustrated above.

See also the numerical simulations Caus2.exe and/or Caustics.exe !!!
2. Gravitational lenses

6. GL MODELS:

6.1. Axially symmetric lens models:

Ray tracing diagram for the uniform disk lens model.

6.1.5. The uniform disk lens model:

A transparent circular disk of matter, seen face on, characterized by a uniform surface mass density \( \Sigma_0 \) has an effective deflecting mass equal to \( \pi \xi^2 \Sigma_0 \), where \( \xi \) represents the impact parameter of a chosen light ray. The deflection angle is thus (see Eq. (4.5)) given by

\[
\alpha(\xi) = 4\pi G \Sigma_0 \xi / c^2. \tag{6.29}
\]

The disk is therefore acting as a normal converging lens (see the ray tracing diagram in the above figure), whose focal length is

\[
f = \xi / \alpha(\xi) = c^2 / (4\pi G \Sigma_0). \tag{6.30}
\]

Note that for a too small value of \( \Sigma_0 \) the focal length \( f \) may turn out to be larger than the size of the Universe.

It is then easy to show that the lens equation (4.3) leads to the solution

\[
\theta = \theta_s / (1 - \kappa), \tag{6.31}
\]

where \( \kappa = \Sigma_0 / \Sigma_c \), \( \Sigma_c \) being the critical surface mass density defined in Eq. (6.5).
2. Gravitational lenses

6. GL MODELS:

6.1. Axially symmetric lens models:

\[ \alpha(\xi) = 4\pi G \Sigma_0 \xi / c^2. \]  \hspace{1cm} (6.29)

\[ f = c^2 / (4\pi G \Sigma_0). \]  \hspace{1cm} (6.30)

\[ \theta = \theta_\text{s} / (1 - \kappa), \]  \hspace{1cm} (6.31)

\[ \kappa = \Sigma_0 / \Sigma_\text{c}, \]

\[ \mu_\text{A} = 1 / (1 - \kappa)^2. \]  \hspace{1cm} (6.32)

Combined bending angle diagram (b) and resulting lensed images produced for the circular source S by two different uniform disk lens models (\(\kappa < 1\) in (a) and \(\kappa > 1\) in (c); see text).

2. Gravitational Lenses:

6. GRAVITATIONAL LENS MODELS:

6.1. Axially symmetric lens models:

6.1.5. The uniform disk lens model:

With the exception of a hypothetical observer that would be precisely located at the focal point (\(\kappa = 1\), the perceived image would then be a fully illuminated disk of light!), the observer just sees one single image of the distant source, somewhat displaced and (de)magnified (see the combined bending angle diagram in the above figure). The magnification of the lensed image is directly found to be

\[ \mu_\text{A} = 1 / (1 - \kappa)^2. \]  \hspace{1cm} (6.32)

Let us note that for an observer located between the lens and the focal point we have \(\kappa < 1\) (cf. \(O_1\) in the corresponding figures), whereas we have \(\kappa > 1\) when the observer is located behind the focal point (cf. \(O_2\)).
2. Gravitational Lenses:

6. GRAVITATIONAL LENS MODELS:
6.1. Axially symmetric lens models:

6.1.6. The truncated uniform disk lens model:

Let us finally consider the case of a truncated uniform disk, i.e. such that we may possibly have \( \xi > R \). For those external rays, the disk effectively acts as a point mass and the lens equation leads to a combination of the solutions (6.9) and/or (6.31), depending on whether the condition \( \theta > R / D_{od} \) or \( \theta < R / D_{od} \) is fulfilled. From the ray tracing diagram depicted in the above figure and the combined bending angle diagrams represented on the next composite figure, we see that only one image can be formed for \( \kappa < 1 \) (cf. \( O_1 \) on the next figures), whereas for \( \kappa > 1 \), there may result the formation of one or three images (cf. \( O_2 \) and \( O_3 \)).
2. Gravitational Lenses:
6. GRAVITATIONAL LENS MODELS:
6.1. Axially symmetric lens models:
6.1.5. The truncated uniform disk lens model:

Combined bending angle diagrams (b and c) and resulting lensed images produced for the circular source S by a truncated uniform disk lens model (a, d and e; see text).